Stability Analysis of Injection-Locked Oscillators in Their Fundamental Mode of Operation

ENRICO F. CALANDRA, MEMBER, IEEE, AND ANTONINO M. SOMMARIVA

Abstract — **Phase-lock stability of fundamental-wave injection-synchronized oscillators is investigated on the basis of a new time-domain approach. Starting from a quite general oscillator modeling and assuming single-frequency quasi-static operation, both exact and first-order approximate stability criteria are derived in a fully analytical form suitable for computer implementation. The examples worked out demonstrate good agreement of this theory with experimental observations available in the literature on multiple-tuned oscillators, whose behavior under large-signal injection was so far predictable only through graphical methods.**

I. Introduction

THE LOCKING STABILITY of injection-synchronized oscillators in their fundamental mode of operation, first analyzed by Van der Pol in 1927 [1], has been investigated by several authors under various simplifying hypotheses [2]–[8]. The simplest approach was presented by Adler [2] who, studying the regenerative triode oscillator, derived an expression of the available locking bandwidth for highly saturated single-tuned oscillators under low injection level operation. Later an extension of this theory to higher levels has been developed by Paciorek [3]. Both of these treatments have practical interest for microwave techniques since they permit simple explanation of a number of experimental observations. Nevertheless, they become inadequate when an instantaneous amplitude limitation cannot be assumed since, in this case, the nonlinear characteristics of the active device cannot be neglected. A more comprehensive circuit model, in which these nonlinearities are taken into account under the simplifying assumption of frequency independence, has been then examined by Kurokawa [4]. By means of a first-order approximation he derived some general stability criteria for low-level injection. This approach has been subsequently extended by other authors to the case in which the nonlinear immittance of the active device is frequency dependent and the level of the injected signal is high [5], [6]. However, as pointed out by Kurokawa himself [7], all these first-order theories fail when the tank circuit of the system under examination exhibits a loop in its immittance locus (such as in multiple-tuned oscillators). As an alternative, he worked out a quasi-static theory capable of explaining all the common phenomena observed in practice [7], [8]. Unfortunately, the proposed method is rather cumbersome, due to the graphical nature of the procedure.

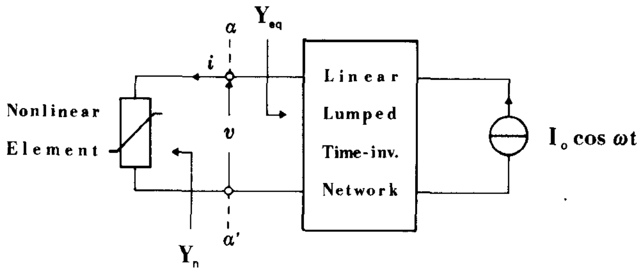


Fig. 1. Equivalent circuit of an injection-locked oscillator.

In this paper, a new quasi-static theory is developed. It permits, whatever the injection level, an exact investigation of the locking stability in a fully analytical form suitable for computer implementation. Starting from a quite general oscillator modeling, a differential equation is first derived, rigorously describing the behavior of the nonlinear system (Section II). A stability-equivalent shortened equation is then obtained which enables a compact formulation of the phase-lock conditions through classical algorithms of System Theory (Section III). On this basis, improved first-order criteria are also deduced allowing a significant comparison with the aforementioned approximate theories (Section IV). As an example, a double-tuned oscillator is analyzed and the results obtained are discussed with reference to other theoretical and experimental data available in the literature (Section V).

II. Equivalent Circuit and Basic Equations

The equivalent circuit employed in our analysis for the modeling of a negative conductance injection-locked microwave oscillator is shown in Fig. 1. In this circuit, three basic elements are in evidence: a sinusoidal current source; a linear, lumped, time-invariant two-port; a nonlinear element. The current source represents the synchronizing generator which injects power, in the fundamental mode

Manuscript received April 6, 1981; revised June 29, 1981. This work was supported by the Progetto Finalizzato "Aiuti alla Navigazione e Controllo del Traffico Aereo" of the Italian National Research Council (CNR).

The authors are with Centro per la Ricerca Elettronicca in Sicilia (CRES), 90046 Monreale, Italy.

0018-9480/81/1100-1137$00.75 ©1981 IEEE

analyzed here, at a frequency ω close to the free-running oscillation frequency. The linear two-port represents all the passive components of the physical system (the resonant cavities, the nonreciprocal coupling network, the load, etc.) including the parasitics due to the package and mounting of the active device [9]. This implies that the α–α' reference plane shown in Fig. 1 does not generally coincide with the terminals of the encapsulated component and, therefore, it may not be physically accessible. The active part of the negative conductance device is represented by the nonlinear one-port, whose differential description, in terms of instantaneous voltage and current, is supposed to be of the form

$$\sum_{0}^{P}{}_{p} D^p\left(\sum_{1}^{K}{}_{k}\alpha_{p,k}v^k\right)=\sum_{0}^{Q}{}_{q}\beta_q D^q i \tag{1}$$

where D stands for the differential operator d/dt, and $\alpha_{p,k}$ and β_q are constants depending on the particular device and bias. The proposed description is a generalization of the instantaneous relationship $i=f(v)$ usually employed for the characterization of memoryless voltage-controlled nonlinear one-ports. Owing to its differential nature, it is suitable for the time-domain modeling of a number of microwave components whose large-signal characteristics are both voltage and frequency dependent (e.g., voltage-excited IMPATT diodes [10]). On the other hand, if the dynamic behavior of the active element is better described by functions nonlinear with current rather than voltage (e.g., current-excited IMPATT diodes [11]), the alternative characterization obtained interchanging voltage and current in (1) can be employed. A dual treatment could then be developed in a manner analogous to that described here.

In order to perform an analytical investigation of the locking stability we have first to derive a differential equation relating, in a suitable form, the voltage across the active element to the injection current. Since the equivalent circuit of Fig. 1 is supposed to represent a monochromatic oscillator entrained by a nearly synchronous driving source, we are allowed to assume, for the RF voltage during transients, the expression

$$v(t)=V(t)\cos(\omega t+\phi(t)) \tag{2}$$

where $V(t)$ and $\phi(t)$ are slowly varying functions over the RF period. This single-frequency quasi-static hypothesis, extensively adopted in the literature dealing with this subject [4]–[8], is usually verified to a high degree of accuracy when untuned-harmonics microwave oscillators are concerned. Therefore, it does not constitute a significant limitation to our purpose.

Substituting (2) into (1), after some manipulation, we get

$$\sum_{0}^{P}{}_{p} D^p\left(\sum_{0}^{K}{}_{k}\left[\hat{\alpha}_{p,k}(V)\cdot V\cos(k\omega t+k\phi)\right]\right)=\sum_{0}^{Q}{}_{q}\beta_q D^q i \tag{3}$$

where the functions $\hat{\alpha}_{p,k}(V)$ are given by

$$\hat{\alpha}_{p,k}(V)\equiv\begin{cases}\displaystyle\sum_{1}^{\mathrm{int}\{K/2\}}{}_{i}\frac{C_{2i}^{i}}{2^{2i}}\alpha_{p,2i}V^{2i-1}, & k=0\\ \displaystyle\sum_{0}^{\mathrm{int}\{(K-k)/2\}}{}_{i}\frac{C_{2i+k}^{i}}{2^{2i+k-1}}\alpha_{p,2i+k}V^{2i+k-1}, & k\neq 0\end{cases} \tag{4}$$

in which C_m^n represents the binomial coefficient $m!/(n!(m-n)!)$. Since, for each k, the terms contained in the square brackets at the left-hand side of (3) are substantially sinusoidal quantities of frequency $k\omega$, we can separate the equations pertaining to different current components through a harmonic balance method. The fundamental-wave differential characterization of the nonlinear element is, therefore, expressed, in complex form, by

$$\sum_{0}^{P}{}_{p} D^p\left(\hat{\alpha}_{p,1}Ve^{j\phi}e^{j\omega t}\right)=\sum_{0}^{Q}{}_{q}\beta_q D^q\left(Ie^{j\psi}e^{j\omega t}\right) \tag{5}$$

where $I(t)$ and $\psi(t)$ are the amplitude and phase of the first-harmonic component of the current. Setting $V(t)$, $\phi(t)$, $I(t)$, $\psi(t)$ constant in this equation, we find the following expression for the nonlinear admittance:

$$Y_n(j\omega,V)\equiv\frac{Ie^{j\psi}}{Ve^{j\phi}}=\frac{\sum_{0}^{P}{}_{p}(j\omega)^p\hat{\alpha}^{p,1}}{\sum_{0\ q}^{Q}{}_{q}(j\omega)^q\beta_q} \tag{6}$$

whose structure makes it evident that the coefficients $\hat{\alpha}_{p,1}$ and β_q can be calculated through computer reduction of steady-state measurements on the active device [12]. From the right-hand side part of the equivalent circuit one may easily obtain, through classical methods of analysis, the differential equation

$$\sum_{0}^{H}{}_{h}\gamma_h D^h\left(Ve^{j\phi}e^{j\omega t}\right)=\sum_{0}^{R}{}_{r}\delta_r D^r\left(Ie^{j\psi}e^{j\omega t}\right)+\sum_{0}^{S}{}_{s}\epsilon_s D^s\left(I_0e^{j\omega t}\right) \tag{7}$$

in which γ_h, δ_r, and ϵ_s are constants depending on the particular network involved. By means of an elimination algorithm (5) and (7) can be combined into

$$\sum_{0}^{N}{}_{n} D^n\left(\mathfrak{N}_n(V)\cdot Ve^{j\phi}e^{j\omega t}\right)=\sum_{0}^{M}{}_{m}(j\omega)^m\mathfrak{M}_m I_0e^{j\omega t}. \tag{8}$$

Introducing now, for the sake of compactness, the complex frequency

$$s\equiv\frac{1}{V}\frac{dV}{dt}+j\left(\omega+\frac{d\phi}{dt}\right) \tag{9}$$

and the differential operator

$$\begin{aligned}&(D+s)^0f(V)\equiv f(V)\\&(D+s)^kf(V)\equiv(D+s)\left\{(D+s)^{k-1}f(V)\right\}\end{aligned} \tag{10}$$

the nth term on the left-hand side of (8) can be put in the form

$$D^n\left(\mathcal{N}_n(V)\cdot V(t)e^{j\phi(t)}e^{j\omega t}\right)=\left((D+s)^n\mathcal{N}_n(V)\right)\cdot V(t)e^{j\phi(t)}e^{j\omega t} \quad (11)$$

which constitutes an extension of the symbolic notation presented by Kurokawa in [4]. Substituting (11) into (8) and canceling the common factor $e^{j\omega t}$, we get finally

$$\left(\sum_{0}^{N}{}_n (D+s)^n\mathcal{N}_n\right)Ve^{j\phi}=\left(\sum_{0}^{M}{}_m (j\omega)^m\mathcal{M}_m\right)I_0. \quad (12)$$

This phasorial relationship, to the authors' knowledge not previously derived, rigorously describes the dynamic behavior of the system under the assumptions made and is the starting point of our stability investigation. Under sinusoidal operation it reduces to the customary form steady-state equation

$$\left(\sum_{0}^{N}{}_n (j\omega)^n\mathcal{N}_n\right)Ve^{j\phi}=\left(\sum_{0}^{M}{}_m (j\omega)^m\mathcal{M}_m\right)I_0 \quad (13)$$

which implicitly defines the coordinates V_s, ϕ_s of the equilibrium points corresponding to a given couple of input parameters I_0, ω.

III. Locking Stability Analysis

In order to evaluate if a particular equilibrium point represents a stable locking condition, we have to ascertain whether a small perturbation in the corresponding steady-state values of amplitude and phase decays or not with time. It will be shown here that such investigation can be performed, after appropriate manipulation of (12), resorting to the classical methods of System Theory. In this connection, let us observe that in the expansion of $(D+s)^n\mathcal{N}_n$ all the terms of the form

$$g(V)\cdot\prod_{k=1}^{n-1}\left(\frac{d^kV}{dt^k}\right)^{p_k}\cdot\left(\frac{d^k\phi}{dt^k}\right)^{q_k} \quad (14)$$

with

$$\sum_{1}^{n-1}{}_k (p_k+q_k)>1$$

can be eliminated since, as will be clear later on, they give no contribution to stability calculations. Therefore, the following reduced expansion holds:

$$(D+s)^n\mathcal{N}_n=(j\omega)^n\mathcal{N}_n+\sum_{1}^{n}{}_k C_n^k\cdot(j\omega)^{n-k}\cdot\left[\left(\frac{\mathcal{N}_n}{V}+\frac{d\mathcal{N}_n}{dV}\right)\frac{d^kV}{dt^k}+j\mathcal{N}_n\frac{d^k\phi}{dt^k}\right]. \quad (15)$$

Substituting the latter into (12) and interchanging the order of the summations at the left-hand side yields

$$\left(\sum_{0}^{N}{}_n (j\omega)^n\mathcal{N}_n+\sum_{1}^{N}{}_k\sum_{k}^{N}{}_n C_n^k\cdot(j\omega)^{n-k}\left[\left(\frac{\mathcal{N}_n}{V}+\frac{d\mathcal{N}_n}{dV}\right)\frac{d^kV}{dt^k}+j\mathcal{N}_n\frac{d^k\phi}{dt^k}\right]\right)Ve^{j\phi}=\left(\sum_{0}^{M}{}_m (j\omega)^m\mathcal{M}_m\right)I_0. \quad (16)$$

If we introduce now

$$\mathcal{N}\equiv\sum_{0}^{N}{}_n (j\omega)^n\mathcal{N}_n$$
$$\mathcal{M}\equiv\sum_{0}^{M}{}_m (j\omega)^m\mathcal{M}_m \quad (17)$$

and recognize that

$$\frac{\partial^k\mathcal{N}}{\partial\omega^k}=j^kk!\sum_{k}^{N}{}_n C_n^k\cdot(j\omega)^{n-k}\mathcal{N}_n \quad (18)$$

the more concise expression is obtained from (16)

$$\left(\mathcal{N}+\sum_{1}^{N}{}_k\frac{j^{-k}}{k!}\left[\left(\frac{1}{V}\frac{\partial^k\mathcal{N}}{\partial\omega^k}+\frac{\partial^{k+1}\mathcal{N}}{\partial\omega^k\partial V}\right)\frac{d^kV}{dt^k}+j\frac{\partial^k\mathcal{N}}{\partial\omega^k}\frac{d^k\phi}{dt^k}\right]\right)Ve^{j\phi}=\mathcal{M}I_0. \quad (19)$$

Separating real and imaginary parts, we get then

$$\mathcal{N}'+\sum_{1}^{\mathrm{int}\{N/2\}}{}_p\frac{(-1)^p}{(2p)!}\left[\left(\frac{1}{V}\frac{\partial^{2p}\mathcal{N}'}{\partial\omega^{2p}}+\frac{\partial^{2p+1}\mathcal{N}'}{\partial\omega^{2p}\partial V}\right)\frac{d^{2p}V}{dt^{2p}}-\frac{\partial^{2p}\mathcal{N}''}{\partial\omega^{2p}}\frac{d^{2p}\phi}{dt^{2p}}\right]-\sum_{1}^{\mathrm{int}\{(N+1/2)\}}{}_q\frac{(-1)^q}{(2q-1)!}\left[\left(\frac{1}{V}\frac{\partial^{2q-1}\mathcal{N}''}{\partial\omega^{2q-1}}+\frac{\partial^{2q}\mathcal{N}''}{\partial^{2q-1}\partial V}\right)\frac{d^{2q-1}V}{dt^{2q-1}}+\frac{\partial^{2q-1}\mathcal{N}'}{\partial\omega^{2q-1}}\frac{d^{2q-1}\phi}{dt^{2q-1}}\right]=\frac{I_0}{V}(\mathcal{M}'\cos\phi+\mathcal{M}''\sin\phi) \quad (20a)$$

and

$$\mathcal{N}''+\sum_{1}^{\mathrm{int}\{N/2\}}{}_p\frac{(-1)^p}{(2p)!}\left[\left(\frac{1}{V}\frac{\partial^{2p}\mathcal{N}''}{\partial\omega^{2p}}+\frac{\partial^{2p+1}\mathcal{N}''}{\partial\omega^{2p}\partial V}\right)\frac{d^{2p}V}{dt^{2p}}+\frac{\partial^{2p}\mathcal{N}'}{\partial\omega^{2p}}\frac{d^{2p}\phi}{dt^{2p}}\right]+\sum_{1}^{\mathrm{int}\{(N+1/2)\}}{}_q\frac{(-1)^q}{(2q-1)!}\left[\left(\frac{1}{V}\frac{\partial^{2q-1}\mathcal{N}'}{\partial\omega^{2q-1}}+\frac{\partial^{2q}\mathcal{N}'}{\partial\omega^{2q-1}\partial V}\right)\frac{d^{2q-1}V}{dt^{2q-1}}-\frac{\partial^{2q-1}\mathcal{N}''}{\partial\omega^{2q-1}}\frac{d^{2q-1}\phi}{dt^{2q-1}}\right]=\frac{I_0}{V}(\mathcal{M}''\cos\phi-\mathcal{M}'\sin\phi) \quad (20b)$$

where

$$\mathcal{N}'\equiv\mathrm{Re}\{\mathcal{N}\}\qquad\mathcal{N}''\equiv\mathrm{Im}\{\mathcal{N}\}$$
$$\mathcal{M}'\equiv\mathrm{Re}\{\mathcal{M}\}\qquad\mathcal{M}''\equiv\mathrm{Im}\{\mathcal{M}\}. \quad (21)$$

As (20a) and (20b) can always be solved for the maximum order derivatives d^NV/dt^N and $d^N\phi/dt^N$, respectively, they can be substituted by an equivalent set of first-order differential equations of the form

$$\left\{\frac{dx_k}{dt}=x_{k+2}\right\}, \qquad k=1,2,\cdots,2N-2$$
$$\frac{dx_{2N-1}}{dt}=f_1(x_1,x_2,\cdots,x_{2N})$$
$$\frac{dx_{2N}}{dt}=f_2(x_1,x_2,\cdots,x_{2N}) \tag{22}$$

with

$$x_k\equiv\begin{cases}\dfrac{d^{\frac{k-1}{2}}V}{dt^{\frac{k-1}{2}}}, & k \text{ odd}\\ \dfrac{d^{\frac{k-2}{2}}\phi}{dt^{\frac{k-2}{2}}}, & k \text{ even.}\end{cases} \tag{23}$$

This fact enables us to make use of the assessed techniques employed for the local stability analysis of nonlinear systems described by state equations [13]. As a first step in this procedure, the Jacobian matrix associated with the system (22) must be calculated

$$J\equiv \begin{bmatrix}\frac{\partial}{\partial x_1}\left(\frac{dx_1}{dt}\right)\cdots\frac{\partial}{\partial x_{2N}}\left(\frac{dx_1}{dt}\right)\\ \cdots\cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\cdots\\ \frac{\partial}{\partial x_1}\left(\frac{dx_{2N}}{dt}\right)\cdots\frac{\partial}{\partial x_{2N}}\left(\frac{dx_{2N}}{dt}\right)\end{bmatrix}_{x_1=V_s,x_2=\phi_s,x_i=0(i=3,\ldots 2N)} \tag{24}$$

According to (24), from (13), (20) and (22), we find

$$J=\left[\begin{array}{c|c} \mathbf{0} & \mathbf{1}\\ \hline a_0 b_0 & a_1\ b_1\cdots a_{N-1} b_{N-1}\\ c_0 d_0 & c_1\ d_1\cdots c_{N-1} d_{N-1}\end{array}\right]_{2N\times 2N} \tag{25}$$

where

$$a_k=\begin{cases}-\dfrac{1}{H_V}\left[\dfrac{\mathfrak{N}'}{V}+\dfrac{\partial\mathfrak{N}'}{\partial V}\right]_{V_s}, & k=0\\ \dfrac{(-1)^{(k+1)/2}}{k!H_V}\left[\dfrac{1}{V}\dfrac{\partial^k\mathfrak{N}''}{\partial\omega^k}+\dfrac{\partial^{k+1}\mathfrak{N}''}{\partial\omega^k,\ldots,V}\right]_{V_s}, & k \text{ odd}\\ -\dfrac{(-1)^{k/2}}{k!H_V}\left[\dfrac{1}{V}\dfrac{\partial^k\mathfrak{N}'}{\partial\omega^k}+\dfrac{\partial^{k+1}\mathfrak{N}'}{\partial\omega^k\partial V}\right]_{V_s}, & k \text{ even}\end{cases}$$

$$b_k=\begin{cases}\dfrac{1}{H_V}[\mathfrak{N}'']_{V_s}, & k=0\\ \dfrac{(-1)^{(k+1)/2}}{k!H_V}\left[\dfrac{\partial^k\mathfrak{N}'}{\partial\omega^k}\right]_{V_s}, & k \text{ odd}\\ \dfrac{(-1)^{k/2}}{k!H_V}\left[\dfrac{\partial^k\mathfrak{N}''}{\partial\omega^k}\right]_{V_s}, & k \text{ even}\end{cases}$$

$$c_k=\begin{cases}-\dfrac{1}{H_\phi}\left[\dfrac{\mathfrak{N}''}{V}+\dfrac{\partial\mathfrak{N}''}{\partial V}\right]_{V_s}, & k=0\\ -\dfrac{(-1)^{(k+1)/2}}{k!H_\phi}\left[\dfrac{1}{V}\dfrac{\partial^k\mathfrak{N}'}{\partial\omega^k}+\dfrac{\partial^{k+1}\mathfrak{N}'}{\partial\omega^k\partial V}\right]_{V_s}, & k \text{ odd}\\ -\dfrac{(-1)^{k/2}}{k!H_\phi}\left[\dfrac{1}{V}\dfrac{\partial^k\mathfrak{N}''}{\partial\omega^k}+\dfrac{\partial^{k+1}\mathfrak{N}''}{\partial\omega^k\partial V}\right]_{V_s}, & k \text{ even}\end{cases}$$

$$d_k=\begin{cases}-\dfrac{1}{H_\phi}[\mathfrak{N}']_{V_s}, & k=0\\ \dfrac{(-1)^{(k+1)/2}}{k!H_\phi}\left[\dfrac{\partial^k\mathfrak{N}''}{\partial\omega^k}\right]_{V_s}, & k \text{ odd}\\ -\dfrac{(-1)^{k/2}}{k!H_\phi}\left[\dfrac{\partial^k\mathfrak{N}'}{\partial\omega^k}\right]_{V_s}, & k \text{ even}\end{cases} \tag{26}$$

with

$$H_V\equiv\frac{(-1)^{\text{int}\{(N+1)/2\}}}{N!}\left[\frac{1}{V}\frac{\partial^N(\mathfrak{N}'-\mathfrak{N}'')}{\partial\omega^N}+\frac{\partial^{N+1}(\mathfrak{N}'-\mathfrak{N}'')}{\partial\omega^N\partial V}\right]_{V_s}$$
$$H_\phi\equiv\frac{(-1)^{\text{int}\{(N+1)/2\}}}{N!}\left[\frac{\partial^N(\mathfrak{N}'-\mathfrak{N}'')}{\partial\omega^N}\right]_{V_s}. \tag{27}$$

The characteristic equation associated with the Jacobian matrix (25) is, therefore, given by

$$\lambda^{2N}-(a_{N-1}+d_{N-1})\lambda^{2N-1}+\sum_{0}^{N-2}{}_k\left(\sum_{k+1}^{N-1}{}_h\, a_h d_{N+k-h}-b_h c_{N+k-h}-a_k-d_k\right)\lambda^{N+k}+\sum_{0}^{N-1}{}_k\left(\sum_{0}^{k}{}_h\, a_h d_{k-h}-b_h c_{k-h}\right)\lambda^k=\sum_{0}^{2N}{}_k\, w_k\lambda^k=0 \tag{28}$$

whose roots, for the asymptotic stability of the equilibrium point considered, must have negative real parts. This investigation can be easily accomplished resorting to one of the conventional methods available in the literature, such as Routh criteria, which are suitable for a straightforward computer implementation.

Before concluding let us recognize that if we had employed the complete expansion of $(D+s)^n\mathfrak{N}_n$, the functions f_1 and f_2 in (22) would contain now additional terms of the form

$$h(x_1)\cdot\prod_{k=3}^{2N}x_k^{r_k}, \qquad \text{with } \sum_{3}^{2N}{}_k\, r_k>1. \tag{29}$$

A little thought reveals, however, that these terms do not affect the exactness of the characteristic polynoml, since their individual contribution to the elements of the Jacobian matrix exhibits at least a multiplicative factor x_k ($k \geqslant 3$) and this factor is zero at equilibrium points.

IV. Improved First-Order Criteria

The analysis carried out in the previous section shows that a set of $2N$ inequalities is to be tested in order to perform an exact investigation of the phase-lock stability. However, if only a qualitative picture of the locking phenomena is needed, a more compact set of stability criteria, deduced from a first-order approximation, can be conveniently applied. This approximate approach is based on the simplifying assumption that all higher order time derivatives of the RF voltage amplitude and phase are negligibly small as compared with the first-order ones, allowing us to make use of the reduced expansion

$$(D+s)^n \mathfrak{N}_n = (j\omega)^n \mathfrak{N}_n + n(j\omega)^{n-1} \cdot \left[\left(\frac{\mathfrak{N}_n}{V} + \frac{\partial \mathfrak{N}_n}{\partial V}\right)\frac{dV}{dt} + j\mathfrak{N}_n \frac{d\phi}{dt}\right]. \quad (30)$$

Substituting (30 into (12) we get, through a procedure similar to that followed in Section III, the system of differential equations

$$\mathfrak{N}' + \left(\frac{1}{V}\frac{\partial \mathfrak{N}''}{\partial \omega} + \frac{\partial^2 \mathfrak{N}''}{\partial \omega \partial V}\right)\frac{dV}{dt} + \frac{\partial \mathfrak{N}'}{\partial \omega}\frac{d\phi}{dt} = \frac{I_0}{V}(\mathfrak{N}'\cos\phi + \mathfrak{N}''\sin\phi)$$

$$\cdot\ \mathfrak{N}'' - \left(\frac{1}{V}\frac{\partial \mathfrak{N}'}{\partial \omega} + \frac{\partial^2 \mathfrak{N}'}{\partial \omega \partial V}\right)\frac{dV}{dt} + \frac{\partial \mathfrak{N}''}{\partial \omega}\frac{d\phi}{dt} = \frac{I_0}{V}(\mathfrak{N}''\cos\phi - \mathfrak{N}'\sin\phi). \quad (31)$$

With this system, a 2×2 Jacobian matrix is associated, whose characteristic polynomial is

$$\lambda^2 + \frac{\Theta}{\Xi}\lambda + \frac{\Delta}{\Xi} \quad (32)$$

where

$$\Delta \equiv \left[\mathfrak{N}'\left(\frac{\mathfrak{N}'}{V} + \frac{\partial \mathfrak{N}'}{\partial V}\right) + \mathfrak{N}''\left(\frac{\mathfrak{N}''}{V} + \frac{\partial \mathfrak{N}''}{\partial V}\right)\right]_{V_s}$$

$$\Theta \equiv \left[\frac{\partial \mathfrak{N}'}{\partial V}\frac{\partial \mathfrak{N}''}{\partial \omega} - \frac{\partial \mathfrak{N}''}{\partial V}\frac{\partial \mathfrak{N}'}{\partial \omega} + \mathfrak{N}'\left(\frac{2}{V}\frac{\partial \mathfrak{N}''}{\partial \omega} + \frac{\partial^2 \mathfrak{N}''}{\partial \omega \partial V}\right) - \mathfrak{N}''\left(\frac{2}{V}\frac{\partial \mathfrak{N}'}{\partial \omega} + \frac{\partial^2 \mathfrak{N}'}{\partial \omega \partial V}\right)\right]_{V_s}$$

$$\Xi \equiv \left[\frac{\partial \mathfrak{N}'}{\partial \omega}\left(\frac{1}{V}\frac{\partial \mathfrak{N}'}{\partial \omega} + \frac{\partial^2 \mathfrak{N}'}{\partial \omega \partial V}\right) + \frac{\partial \mathfrak{N}''}{\partial \omega}\left(\frac{1}{V}\frac{\partial \mathfrak{N}''}{\partial \omega} + \frac{\partial^2 \mathfrak{N}''}{\partial \omega \partial V}\right)\right]_{V_s}. \quad (33)$$

According to this reduced polynomial, the following first-order stability criteria result:

$$\frac{\Delta}{\Xi} > 0 \qquad \frac{\Theta}{\Xi} > 0 \quad (34)$$

to which both the tern (triplet) of conditions $\{\Delta, \Theta, \Xi > 0\}$ and $\{\Delta, \Theta, \Xi < 0\}$ would simultaneously correspond. Actually, a comparison with the exact theory permits one to recognize that, for each equilibrium point, only one of these terns must be taken into account, the sign ambiguity being merely a consequence of the approximations made. In fact, it is easily verified that the coefficients w_0 and w_1 of the exact polynomial are related to the quantities Δ and Θ through the expressions

$$w_0 = \frac{\Delta}{H} \qquad w_1 = \frac{\Theta}{H} \quad (35)$$

where

$$H \equiv H_V \cdot H_\phi. \quad (36)$$

Since, for stable locking, these coefficients are required to be positive, it results that the actual stability conditions must be formulated as follows:

$$\Delta > 0;\ \Theta > 0;\ \Xi > 0, \qquad \text{for } H > 0 \quad (37a)$$

$$\Delta < 0;\ \Theta < 0;\ \Xi < 0, \qquad \text{for } H < 0. \quad (37b)$$

From the preceding discussion it turns out that no reliable stability criteria can be provided by a strictly first-order analysis since at least one higher order information, such as the sign of H in this theory, is needed. This circumstance, however, does not greatly affect the convenience of the simplified approach described here, in that the additional calculations involved are of no significant complexity. Moreover, in the most common cases, the sign of H is positive and this fact can be often recognized simply by inspection of the circuit equations.

As pointed out in Section I, first-order criteria by other authors have been already derived, following different procedures, all their results being similar [4]–[6]. The most general treatment is that of Hansson and Lundström, who obtained the conditions

$$\Delta_H > 0 \qquad \Theta_H > 0 \quad (38)$$

with

$$\Delta_H \equiv \left[G_T\left(G_T + \frac{\partial G_T}{\partial V}V\right) + B_T\left(B_T + \frac{\partial B_T}{\partial V}V\right)\right]_{V_s}$$

$$\Theta_H \equiv \left[\left(\frac{\partial G_T}{\partial V}\frac{\partial B_T}{\partial \omega} - \frac{\partial B_T}{\partial V}\frac{\partial G_T}{\partial \omega}\right)V + 2\left(G_T\frac{\partial B_T}{\partial \omega} - B_T\frac{\partial G_T}{\partial \omega}\right)\right]_{V_s} \quad (39)$$

in which G_T and B_T are, respectively, real and imaginary parts of the total admittance $Y_T \equiv Y_n + Y_{eq} = I_{eq}e^{j\theta}/Ve^{j\phi}$.[1] Recognizing that this admittance is related to the function $\mathfrak{N}$ by the expression

$$Y_T(j\omega, V) = \frac{I_{eq}e^{j\theta}}{\mathfrak{N} I_0}\mathfrak{N} = f(j\omega)\cdot \mathfrak{N}(j\omega, V) \quad (40)$$

[1] Y_{eq} and $I_{eq}e^{j\theta}$ are the Norton's equivalent admittance and current of the linear network on the right-hand side of the α–α' reference plane.

it can be demonstrated that the condition $\Delta_H>0$ is actually equivalent to $\Delta>0$, whereas the condition $\Theta_H>0$ does not correspond to $\Theta>0$, except for free-running oscillators or particular circuits. Furthermore, a condition analogous to $\Xi \gtreqless 0$ does not subsist, nor is the influence of a quantity like H taken into account. Owing to these differences, our first-order analysis gives quite satisfactory results even in those cases in which Hansson's theory, while directly applicable ($H>0$), does not succeed (see Section V for a relevant example). This better predictability is due, in the authors' opinion, to the more rigorous approach followed here in deriving the characteristic polynomial of the system, which causes the functions $\mathfrak{N}'$ and $\mathfrak{N}''$ to be handled instead of G_T and B_T. Indeed, it does not seem a correct procedure to deduce a set of variational equations merely by substituting the complex frequency s for $j\omega$ in Y_T and then expanding, as done in [4]–[6].

V. Examples

In this section, the proposed methodology is applied to a double-tuned oscillator modeled by the first-harmonic equivalent circuit depicted in Fig. 2. For the sake of simplicity the active element is supposed to be characterized by a cubic instantaneous i–v relationship. According to (4) and (6), the first-harmonic nonlinear conductance $G_n(V)$ then has the form

$$G_n(V)=G_{n0}+G_{n2}V^2. \tag{41}$$

Assuming a synchronous tuning of the two resonant circuits ($LC=L'C'=1/\omega_0^2$) the following steady-state equation is obtained:

$$\left[\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 QQ'-\left(\frac{\omega}{\omega_0}\right)^2(1+R_0G_n)+j\frac{\omega}{\omega_0}\cdot\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)(Q+Q'R_0G_n)\right]\frac{V}{V_0}e^{j\phi}=-2\left(\frac{\omega}{\omega_0}\right)^2\sqrt{\frac{P_{as}}{P_0}} \tag{42}$$

where

$$P_{as}\equiv\frac{R_0I_0^2}{8} \tag{43}$$

is the available injection power

$$Q\equiv\omega_0CR_0$$
$$Q'\equiv\frac{\omega_0L'}{R_0} \tag{44}$$

are the equivalent quality factors of the resonant circuits, and

$$V_0\equiv\sqrt{-\frac{1+R_0G_{n0}}{R_0G_{n2}}}$$
$$P_0\equiv\frac{V_0^2}{2R_0} \tag{45}$$

are the RF voltage amplitude and output power corresponding to the ω_0 frequency undriven oscillation. Identifying in (42) the normalized expressions of $\mathfrak{N}'$ and $\mathfrak{N}''$

$$\mathfrak{N}'=\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2QQ'-\left(\frac{\omega}{\omega_0}\right)^2(1+R_0G_n)$$
$$\mathfrak{N}''=\frac{\omega}{\omega_0}\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)(Q+Q'R_0G_n) \tag{46}$$

by means of (26) and (28), the eighth-degree characteristic equation of the system is deduced. Since the behavior of a double-tuned oscillator is different depending on whether the two resonant circuits are undercoupled ($Q>Q'$) or

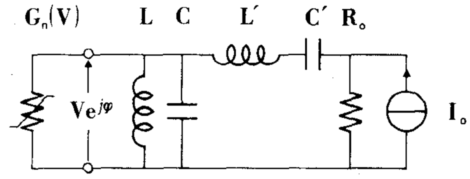


Fig. 2. First-harmonic equivalent circuit of a double-tuned injection-locked oscillator.

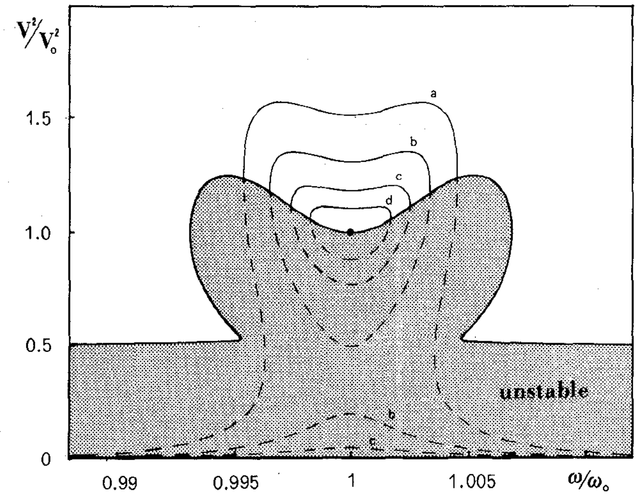


Fig. 3. Stability diagrams (exact theory) for the circuit of Fig. 2: $R_0G_{n0}=-2$, $Q=100$, $Q'=75$. Constant injection power loci are superimposed: (a) $P_{as}/P_0=-10$ dB, (b) -15 dB, (c) -20 dB, (d) -25 dB.

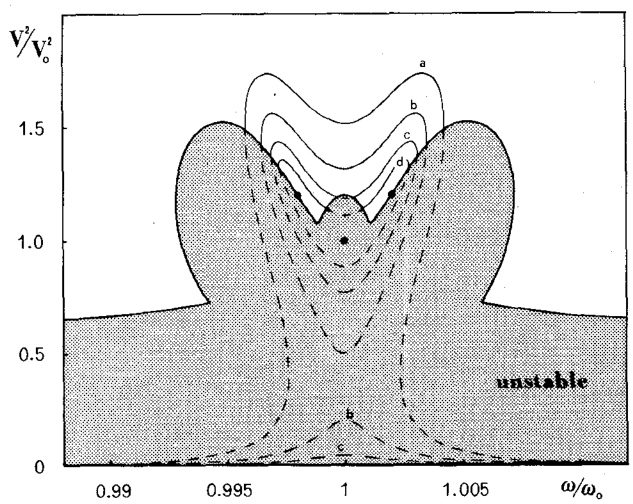


Fig. 4. Stability diagrams (exact theory) for the circuit of Fig. 2: $R_0G_{n0}=-2$, $Q=100$, $Q'=125$. Constant injection power loci are superimposed: (a) P_{as}/P_0-10 dB, (b) -15 dB, (c) -20 dB, (d) -25 dB.

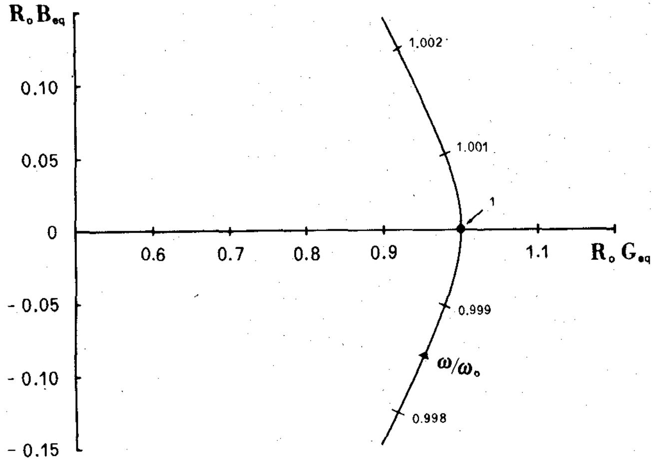


Fig. 5. Admittance locus $Y_{eq}(j\omega)$ for the circuit of Fig. 2: $Q=100$, $Q'=75$.

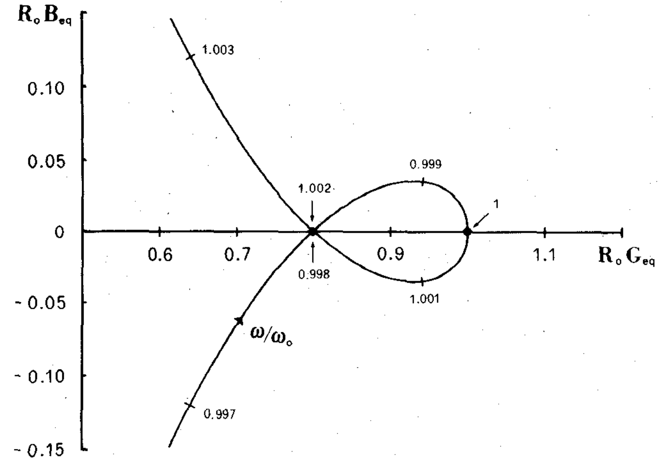


Fig. 6. Admittance locus $Y_{eq}(j\omega)$ for the circuit of Fig. 2: $Q=100$, $Q'=125$.

overcoupled ($Q<Q'$), both cases have been analyzed. The corresponding stability regions, plotted on a voltage–frequency plane by means of a digital computer, are shown in Figs. 3 and 4, respectively. Constant injection power loci, as calculated from (42), are superimposed. When the oscillator is undercoupled, there is only one free-running equilibrium point and it is stable. Around it, the locking band monotonically increases for increasing synchronizing powers. Otherwise, when the oscillator is overcoupled, out of the three free-running equilibrium points only the outer two are stable. Around them, when a synchronizing signal is injected, two separated locking bands arise, which, beyond a certain power level, join together forming a continuous band. Therefore, for sufficiently high injection levels, the phase lock can be achieved also in the neighborhood of the ω_0 frequency, where oscillations are normally inhibited.

All these phenomena are actually observed in practice and have been reported by Kurokawa [8]. He explained the experimental results through his graphical method, correlating this behavior, typical of multiple-tuned oscillators, with the occurrence of a loop in the admittance locus $Y_{eq}(j\omega)$ (cf. Figs. 3 and 4 with Figs. 5 and 6).

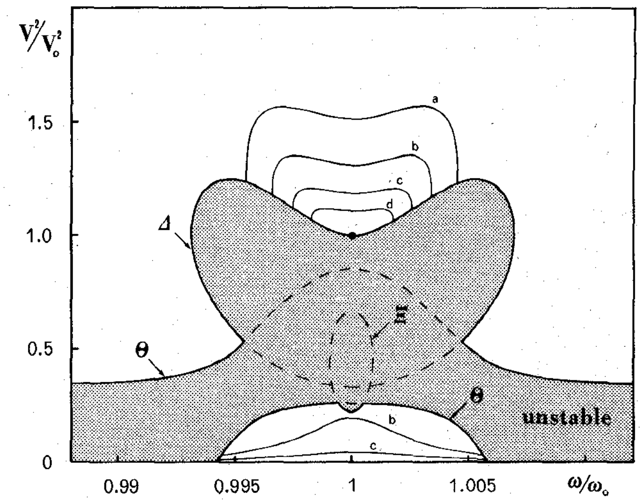


Fig. 7. Stability diagrams (first-order theory) for the circuit of Fig. 2: $R_0G_{n0}=-2$, $Q=100$, $Q'=75$. Constant injection power loci are superimposed: (a) $P_{as}/P_0=-10$ dB, (b) -15 dB, (c) -20 dB, (d) -25 dB.

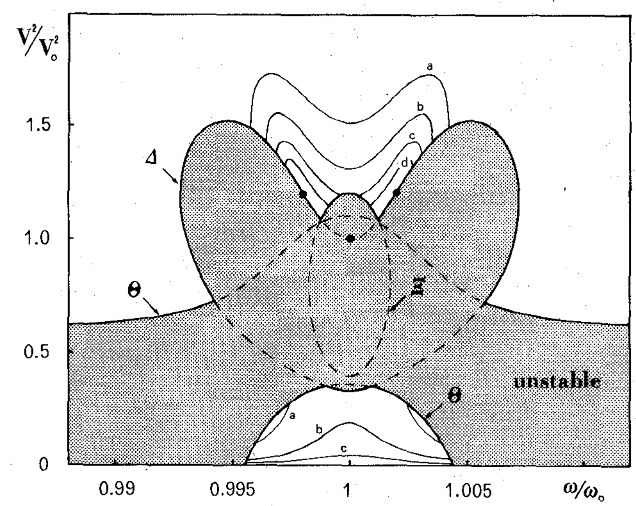


Fig. 8. Stability diagrams (first-order theory) for the circuit of Fig. 2: $R_0G_{n0}=-2$, $Q=100$, $Q'=125$. Constant injection power loci are superimposed: (a) $P_{as}/P_0=-10$ dB, (b) -15 dB, (c) -20 dB, (d) -25 dB.

The diagrams of Figs. 3 and 4 can be now compared with those provided by our first-order theory. Since, for this circuit, the quantity H is always greater than zero, stable areas are identified by positive Δ, Θ, and Ξ. Therefore, drawing the curves defined by $\Delta=0$, $\Theta=0$, and $\Xi=0$, the stability regions depicted in Figs. 7 and 8 are obtained. The comparison shows that, no matter what the shape of the admittance locus involved, these diagrams are nearly indistinguishable from the exact ones in the upper part, whereas in the lower part, due to the approximations made, they exhibit an improper stability region. Consequently, if only a perturbation of the free-running operating conditions is considered (which does not necessarily correspond, however, to low injection levels), the first-order analysis described here is reasonably accurate and sufficient, in any case, for a qualitative investigation of the locking stability.

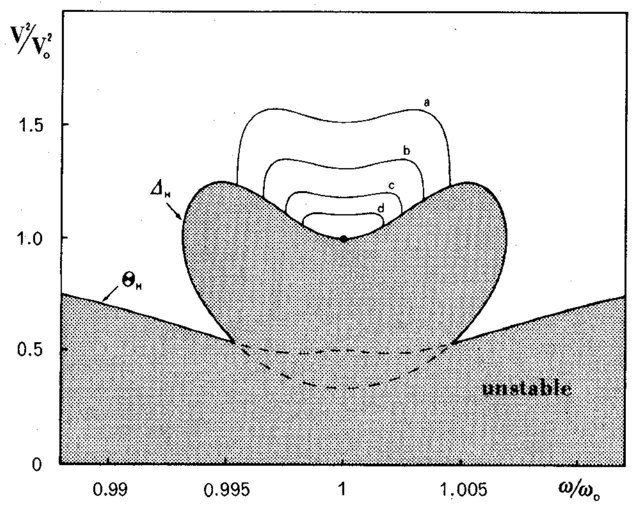


Fig. 9. Stability diagrams (Hansson's theory) for the circuit of Fig. 2: $R_0G_{n0}=-2$, $Q=100$, $Q'=75$. Constant injection power loci are superimposed: (*a*) $P_{as}/P_0=-10$ dB, (*b*) -15 dB, (*c*) -20 dB, (*d*) -25 dB.

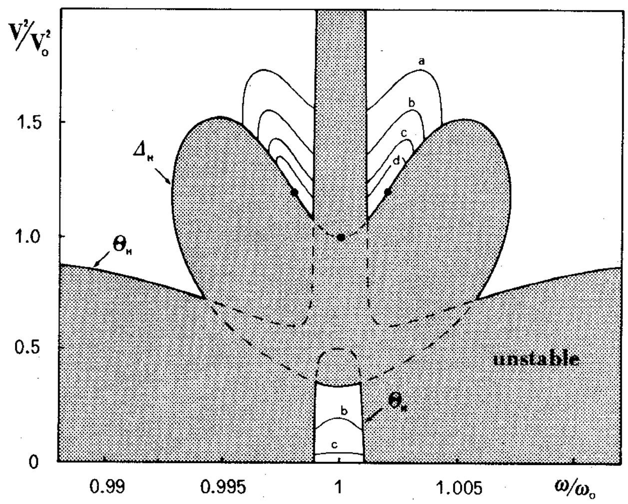


Fig. 10. Stability diagrams (Hansson's theory) for the circuit of Fig. 2: $R_0G_{n0}=-2$, $Q=100$, $Q'=125$. Constant injection power loci are superimposed: (*a*) P_{as}/P_0-10 dB, (*b*) -15 dB, (*c*) -20 dB, (*d*) -25 dB.

On the contrary, less homogeneous results are provided by Hansson's criteria (see Figs. 9 and 10). Indeed, while a good agreement with the exact diagram is found when the oscillator is undercoupled, such an agreement does not exist when the oscillator is overcoupled. Particularly, in this latter case, due to the influence of the condition on Θ_H, a continuous locking band cannot be achieved whatever the injection level, in clear contradiction with experimental observations.

VI. Conclusions

For the assumed circuit characterization of an injection-locked oscillator, both exact and first-order stability conditions have been derived in analytical form. The examples worked out demonstrate good agreement of this theory with experimental observations on multiple-tuned oscillators, whose behavior under large-signal injection was so far predictable only through graphical methods. The main hypothesis involved in our calculations is a single-mode quasi-static operation that is usually verified in microwave sources with untuned harmonics. The proposed approach can then be useful in microwave techniques for performance optimization of injection-synchronized oscillators, especially when computer-aided design procedures are employed.

References

[1] B. Van der Pol, "Forced oscillations in a circuit with nonlinear resistance," *Phil. Mag.*, vol. 3, pp. 65–80, Jan. 1927.

[2] R. Adler, "A study of locking phenomena in oscillators," *Proc. IRE*, vol. 34, pp. 351–357, June 1946; reprinted in *Proc. IEEE*, vol. 61, pp. 1380–1385, Oct. 1973.

[3] L. J. Paciorek, "Injection locking of oscillators," *Proc IEEE*, vol. 53, pp. 1723–1727, Nov. 1965.

[4] K. Kurokawa, "Some basic characteristics of broadband negative resistance oscillator circuits," *Bell Syst. Tech. J.*, vol 48, pp. 1937–1955, July–Aug. 1969.

[5] Y. Takayama, "Power amplification with IMPATT diodes in stable and injection-locked modes," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 266–272, Apr. 1972.

[6] G. H. B. Hansson and K. I. Lundström, "Stability criteria for phase-locked oscillators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 641–645, Oct. 1972.

[7] K. Kurokawa, "Stability of injection-locked oscillators," *Proc. IEEE* (Lett.), vol. 60, pp. 907–908, July 1972.

[8] ——, "Injection locking of microwave solid-state oscillators," *Proc. IEEE*, vol. 61, pp. 1386–1410, Oct. 1973.

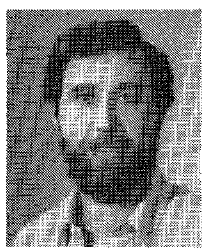
[9] J. W. Monroe, "The effects of package parasitics on the stability of microwave negative resistance devices," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-21, pp. 731–735, Nov. 1973.

[10] D. L. Scharfetter and H. K. Gummel, "Large-signal analysis of a silicon Read diode oscillator," *IEEE Trans. Electron Devices*, vol. ED-16, pp. 64–77, Jan. 1969.

[11] M. S. Gupta and R. J. Lomax, "A current-excited large-signal analysis of IMPATT devices and its circuit implication," *IEEE Trans. Electron Devices*, vol. ED-20, pp. 395–399, Apr. 1973.

[12] J. W. Gewartowski and J. F. Morris, "Active IMPATT diode parameters obtained by computer reduction of experimental data," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 157–161, Mar. 1970.

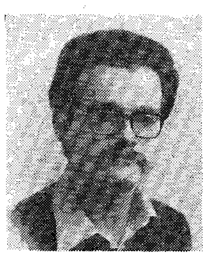
[13] A. Blaquière, *Nonlinear System Analysis*. New York: Academic, 1966.



Enrico F. Calandra (S'76–M'79) was born in Messina, Italy, in 1951. He graduated (with Honors) in electronic engineering from the University of Palermo, Palermo, Italy, in 1978.

After graduating he joined the Research Staff of the Centro per la Ricerca Elettronica in Sicilia (CRES), Monreale, Italy, where he was engaged in the development of coherent MTI radar systems. His research interests are focused on microwave oscillators and amplifiers and low-noise techniques and measurements.

Dr. Calandra is a member of the Associazione Elettrotecnica ed Elettronica Italiana.



Antonino M. Sommariva was born in Palermo, Italy, on October 12, 1953. He received the Doctor Degree in electronic engineering from the University of Palermo, Palermo, Italy, in 1977.

In 1978, he joined the Centro per la Ricerca Elettronica in Sicilia (CRES), Monreale, Italy, as a Member of the Research Staff. In 1979, he was appointed as Head of the Microwave Department of CRES. His present research interests are in the areas of circuit and communication theory.